# NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS USING EULER'S METHOD 

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## Introduction



In this article our objective is to develop a method for approximating the solution of an initial-value problem of the form

$$
y^{\prime}=f(x, y), \quad y(x 0)=y 0
$$

We will not attempt to approximate $\mathrm{y}(\mathrm{x})$ for all values of x ; rather, we will choose some small increment ? x and focus on approximating the values of $y(x)$ at a succession of $x$-values spaced ?x units apart, starting from $x 0$. We will denote these $x$-values by $x_{1}=x_{0}+? \mathrm{x}, \mathrm{x}_{2}=\mathrm{x}_{1}+? \mathrm{x}, \mathrm{x}_{3}=\mathrm{x}_{2}+$ ? $\mathrm{x}, \mathrm{x}_{4}=\mathrm{x}_{3}+$ ? x , $\qquad$ and we will denote the approximations of $\mathrm{y}(\mathrm{x})$ at these points by $\mathrm{y}_{1}$ ? $\mathrm{y}\left(\mathrm{x}_{1}\right)$, y2?y(x2), y3?y(x3)
The technique that we will describe for obtaining these approximations is calledEuler's Method.

## Euler's Method

To approximate the solution of the initial-value problem

$$
y^{\prime}=f(x, y), \quad y(x 0)=y_{0}
$$

Proceed as follows
Step 1: Choose a nonzero number ?x to serve as an increment or step size along the x -axis, and let
$\mathrm{x} 1=\mathrm{x} 0+$ ? $\mathrm{x}, \mathrm{x} 2=\mathrm{x} 1+? \mathrm{x}, \mathrm{x} 3=\mathrm{x} 2+? \mathrm{x}$,
Step 2: Compute successively

$$
\begin{aligned}
& \mathrm{y} 1=\mathrm{y} 0+\mathrm{f}(\mathrm{x} 0, \mathrm{y} 0) ? \mathrm{x} \\
& \mathrm{y} 2=\mathrm{y} 1+\mathrm{f}(\mathrm{x} 1, \mathrm{y} 1) ? \mathrm{x} \\
& \mathrm{y} 3=\mathrm{y} 2+\mathrm{f}(\mathrm{x} 2, \mathrm{y} 2) ? \mathrm{x}
\end{aligned}
$$

$$
y n+1=y n+f(x n, y n) ? x
$$

The numbers $\mathrm{y} 1, \mathrm{y} 2, \mathrm{y} 3$ $\qquad$ in these equations are the approximations of $\mathrm{y}(\mathrm{x} 1), \mathrm{y}(\mathrm{x} 2), \mathrm{y}(\mathrm{x} 3)$,
Example: Use Euler's Method with a step size of 0.1 to make a table of approximate values of the solution of the initial-value problem
$y^{\prime}=y-x, y(0)=2$ over the interval $0 ? x ? 1$.

Solution: In this problem we have $f(x, y)=y-x, x 0=0$ and $y 0=2$. Moreover, since the step size is 0.1 , the $x$-values at which the approximate values will be obtained are
$\mathrm{x} 1=0.1, \mathrm{x} 2=0.2, \mathrm{x} 3=0.3$, $\mathrm{x} 9=0.9, \mathrm{x} 10=1$

The first three approximations are
$\mathrm{y} 1=\mathrm{y} 0+\mathrm{f}(\mathrm{x} 0, \mathrm{y} 0) ? \mathrm{x}=2+(2-0)(0.1)=2.2$
$\mathrm{y} 2=\mathrm{y} 1+\mathrm{f}(\mathrm{x} 1, \mathrm{y} 1) ? \mathrm{x}=2.2+(2.2-0.1)(0.1)=2.41$
$\mathrm{y} 3=\mathrm{y} 2+\mathrm{f}(\mathrm{x} 2, \mathrm{y} 2) ? \mathrm{x}=2.41+(2.41-0.2)(0.1)=2.631$
Here is a way of organizing all 10 approximations rounded to five decimal places.
EULER's METHOD FOR $y^{\prime}=y-x, y(0)=2$ with $\Delta x=0.1$

| $n$ | $x_{n}$ | $y_{n}$ | $f\left(x_{n}, y_{n}\right) \Delta x$ | $y_{n+1}=y_{n}+f\left(x_{n}, y_{n}\right) \Delta x$ |
| :---: | :---: | :---: | :---: | :---: |
| 0 | 0 | 2.00000 | 0.20000 | 2.20000 |
| 1 | 0.1 | 2.20000 | 0.21000 | 2.41000 |
| 2 | 0.2 | 2.41000 | 0.22100 | 2.63100 |
| 3 | 0.3 | 2.63100 | 0.23310 | 2.86410 |
| 4 | 0.4 | 2.86410 | 0.24641 | 3.11051 |
| 5 | 0.5 | 3.11051 | 0.26105 | 3.37156 |
| 6 | 0.6 | 3.37156 | 0.27716 | 3.64872 |
| 7 | 0.7 | 3.64872 | 0.29487 | 3.94359 |
| 8 | 0.8 | 3.94359 | 0.31436 | 4.25795 |
| 9 | 0.9 | 4.25795 | 0.33579 | 4.59374 |
| 10 | 1.0 | 4.59374 | - | - |

Observe that each entry in the last column becomes the next entry in the third column.

## Accuracy of Euler's Method

We can compare the approximate values of $\mathrm{y}(\mathrm{x})$ produced by Euler's Method with decimal approximation of the exact values. The absolute error is calculated as
Absolute error $=\mid$ exact value - approximation $\mid$ and the percentage error as

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Percentage error \(=\mid\) exact value - approximation \(\mid \times 100 \%\)
|exact value|
```

The absolute error and percentage error of the above problem is shown below.

| x | EXACT | EULER | ABSOLUTE | PERCENTAGE |
| :--- | :---: | :---: | :---: | :---: |
|  | SOLUTION | APPROXIMATION | ERROR | ERROR |
| 0.0 | 2.00000 | 2.00000 | 0.00000 | 0.00 |
| 0.1 | 2.20517 | 2.20000 | 0.00517 | 0.23 |
| 0.2 | 2.42140 | 2.41000 | 0.01140 | 0.47 |
| 0.3 | 2.64986 | 2.63100 | 0.01886 | 0.71 |
| 0.4 | 2.89182 | 2.86410 | 0.02772 | 0.96 |
| 0.5 | 3.14872 | 3.11051 | 0.03821 | 1.21 |
| 0.6 | 3.42212 | 3.37156 | 0.05056 | 1.48 |
| 0.7 | 3.71375 | 3.64872 | 0.06503 | 1.75 |
| 0.8 | 4.02554 | 3.94359 | 0.08195 | 2.04 |
| 0.9 | 4.35960 | 4.25795 | 0.10165 | 2.33 |
| 1.0 | 4.71828 | 4.59374 | 0.12454 | 2.64 |

## Exponential Growth and Decay Models

A quantity $y=y(t)$ is said to be an exponential growth model it is increases at a rate that is proportional to the amount of the quantity present, and it is said to have an exponential decay model if ir decreases at a rate that is proportional to the amount of the quantity present. Thus, for an exponential growth model, the quantity $\mathrm{y}(\mathrm{t})$ satisfies an equation of the form

$$
\mathrm{dy} / \mathrm{dt}=\mathrm{ky}(\mathrm{k}>0)
$$

and for an exponential decay model, the quantity $y(t)$ satisfies an equation of the form

$$
\mathrm{dy} / \mathrm{dt}=-\mathrm{ky}(\mathrm{k}>0)
$$

The constant k is called the growth constant or the decay constant, as appropriate.
The above written two equations are first order linear equations, since they can be rewritten as

$$
(d y / d t)-k y=0 \text { and }(d y / d t)+k y=0
$$

To illustrate how these equations can be solved, suppose that a quantity $y=y(t)$ has an exponential growth model and we know the amount of the quantity at some point in time, $y=y_{0}$ when $t=0$. Thus a general formula for $y(t)$ can be obtained by solving the initial value problem

$$
(\mathrm{dy} / \mathrm{dt})-\mathrm{ky}=0, \mathrm{y}(0)=\mathrm{y} 0
$$

Multiplying by integrating factor, $\mu=e^{-k t}$ yields $d / d t\left(e^{-k t} y\right)=0$ and then integrating with respect to ' $t$ ' yields

$$
\mathrm{e}^{-\mathrm{kt}} \mathrm{y}=\mathrm{C} \text { or } \mathrm{y}=\mathrm{Ce} \mathrm{e}^{\mathrm{kt}}
$$

the initial condition implies that $\mathrm{y}=\mathrm{y} 0$ when $\mathrm{t}=0$, from which it follows that $\mathrm{C}=\mathrm{y} 0$. Thus, the solution of the initial-value problem is

$$
\mathrm{y}=\mathrm{y} 0 \mathrm{e}^{\mathrm{kt}}
$$

We leave it for you to show that if $\mathrm{y}=\mathrm{y}(\mathrm{t})$ has an exponential decay model, and if $\mathrm{y}(0)=\mathrm{y} 0$, then

$$
y=y_{0} e^{-k t}
$$

## Doubling time and half-life

If a quantity $y$ has an exponential growth model, then the time required for the original size to double is called the doubling time, and if $y$ has an exponential decay model, then the time required for the original size to reduce by half is called the half-life . As it turns out, doubling time and half-life depend only on the growth or decay rate and not on the amount present initially. To see why this is so, suppose that $\mathrm{y}=\mathrm{y}(\mathrm{t})$ has an exponential growth model $\mathrm{y}=\mathrm{y}_{0} \mathrm{e}^{\mathrm{kt}}$ and let T denote the amount of time required for y to double in size. Thus, at time $\mathrm{t}=\mathrm{T}$ the value of y will be 2 yo and hence the above equation becomes $2 \mathrm{y} 0=\mathrm{y} 0 \mathrm{e}^{\mathrm{kT}}$ or $\mathrm{e}^{\mathrm{kT}}=2$

Hence $T=(1 / k) \ln 2$
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## Reference Links:

- http://en.wikipedia.org/wiki/Initial_value_problem
- http://en.wikipedia.org/wiki/Euler_method
- http://en.wikipedia.org/wiki/Exponential_growth
- http://en.wikipedia.org/wiki/Half-life


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