## RIEMANN SUM AND TRAPEZOIDAL RULE

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## Riemann Sum Approximation



The definite integral of a continuous function ' f ' over an interval $[\mathrm{a}, \mathrm{b}]$ is computed as ${ }_{a}{ }^{\text {? }}{ }^{b} \mathrm{f}(\mathrm{x}) \mathrm{dx}=\lim ? \mathrm{f}\left(\mathrm{x}_{\mathrm{k}}{ }^{*}\right)$ ? x , where the sum that appears on the right side is called Riemann sum. In this formula, the interval [a,b] is divided into $n$ subintervals of width $? \mathrm{x}=$ (b-a) $/ \mathrm{n}$, and $\mathrm{x}_{\mathrm{k}} *$ denotes an arbitrary point in the $\mathrm{k}^{\text {th }}$ sub-interval It follows that as n increases the Riemann sum will eventually be a good approximation to the integral, which we denote by writing

$$
\begin{aligned}
& \mathrm{a} \text { ? } \mathrm{b}(\mathrm{x}) \mathrm{dx} ? ? \mathrm{f}\left(\mathrm{xk}^{*}\right) ? \mathrm{x} \\
& \mathrm{a} ?^{\mathrm{b}} \mathrm{f}(\mathrm{x}) \mathrm{dx} ? ? \mathrm{x}\left[\mathrm{f}\left(\mathrm{x}_{1}^{*}\right)+\mathrm{f}\left(\mathrm{x}_{2}^{*}\right)+\ldots \ldots .+\mathrm{f}\left(\mathrm{x}_{\mathrm{n}}{ }^{*}\right)\right]
\end{aligned}
$$



Here we denote the values of ' f ' at the endpoints of the subintervals by
$y_{0}=f(a), y_{1}=f\left(x_{1}\right), y_{2}=f\left(x_{2}\right), \ldots \ldots \ldots, y_{n-1}=f\left(x_{n-1}\right), y_{n}=f(b)$ and we will denote the values of $f$ at the midpoints of the subintervals by $y_{m 1}, y_{m}, \ldots \ldots y_{m n}$

## Trapezoidal Approximation

The left-hand and right hand endpoint approximations are rarely used in applications; however, if we take the average of the lefthand and right hand endpoint approximations, we obtain a result, called the trapezoidal approximation, which is commonly used as,

$$
\mathrm{a} ?
$$


represents the area under $f(x)$ over [a, b]. Geometrically, the trapezoidal approximation formula results if we approximate this area by the sum of the trapezoidal areas as shown in the figure

Left end point Approximation: The formula for evaluating left end point approximation is given by

$$
a ?^{b} f(x) d x=(b-a) / n\left[y 0+y_{1}+\ldots \ldots \ldots .+y_{n-1}\right]
$$

Right Endpoint Approximation: The formula for evaluating right end point approximation is given by

$$
a ?^{b} f(x) d x=(b-a) / n[y 1+y 2+\ldots \ldots \ldots \ldots+y n]
$$

Mid-Point Approximation: The formula for evaluating midpoint approximation is given by

$$
a ?^{b} f(x) d x=(b-a) / n\left[y m_{1}+y m_{2}+\ldots \ldots \ldots . . y m n\right] \text { where } m_{1}, m_{2} \ldots \ldots m_{n} \text { represents the mid values. }
$$

Example: Use Trapezoidal rule to approximate 0 ? $\sin \mathrm{x} d x$ using $\mathrm{n}=10$ sub intervals
Solution: $\mathrm{a}=0, \mathrm{~b}=? \mathrm{n}=10$ and $\mathrm{f}(\mathrm{x})=\sin \mathrm{x}, \quad(\mathrm{b}-\mathrm{a}) / \mathrm{n}=? / 10$

| $i$ | 0 | 1 | 2 | 3 | $\ldots \ldots \ldots \ldots \ldots$. | 10 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $x_{i}$ | 0 | $\pi / 10$ | $2 \pi / 10$ | $3 \pi / 10$ | $\ldots \ldots \ldots \ldots \ldots$. | $10 \pi / 10$ |
| $y_{i} \sin (0)$ | $\sin (\pi / 10)$ | $\sin (2 \pi / 10)$ | $\sin (3 \pi / 10)$ | $\ldots \ldots \ldots \ldots \ldots \ldots$. | $\sin (10 \pi / 10)$ |  |

$$
\begin{aligned}
0 ?
\end{aligned} \begin{aligned}
& ? \\
& \sin x \mathrm{dx}=(? / 20)\left[\mathrm{y} 0+2 \mathrm{y} 1+2 \mathrm{y} 2+\ldots \ldots \ldots .+2 \mathrm{yn}-1+\mathrm{yn}^{2}\right] \\
&=(? / 20)[\sin (0)+2 \sin (? / 10)+2 \sin (2 ? / 10)+\ldots \ldots . . \sin (10 ? / 10) \\
&=1.983523538
\end{aligned}
$$

## Comparison of the Midpoint and Trapezoidal Approximations

The table below shows the comparison between midpoint and trapezoidal approximations for the function $\ln 2={ }_{1} ?^{2}(1 / x) d x$ with $\mathrm{n}=10$ subdivisions

Midpoint Approximation

| i | Midpoint $\left(\mathrm{m}_{\mathrm{i}}\right)$ | $\mathrm{y}_{\mathrm{mi}}=\mathrm{f}\left(\mathrm{m}_{\mathrm{i}}\right)=1 / \mathrm{m}_{\mathrm{i}}$ |
| :---: | :---: | :---: |
| 1 | 1.05 | 0.952380952 |
| 2 | 1.15 | 0.869565217 |
| 3 | 1.25 | 0.800000000 |
| 4 | 1.35 | 0.740740741 |
| 5 | 1.45 | 0.689655172 |
| 6 | 1.55 | 0.645161290 |
| 7 | 1.65 | 0.606060606 |
| 8 | 1.75 | 0.571428571 |
| 9 | 1.85 | 0.540540541 |
| 10 | 1.95 | 0.512820513 |

$1_{1} ?^{2}(1 / \mathrm{x}) \mathrm{dx}=(0.1)(6.928353603)=0.692835360$
Trapezoidal Approximation

| i | Endpoint $\left(\mathrm{x}_{\mathrm{i}}\right)$ | $\mathrm{y}_{\mathrm{i}}=\mathrm{f}\left(\mathrm{x}_{\mathrm{i}}\right)=1 / \mathrm{x}_{\mathrm{i}}$ | Multiplier $\left(\mathrm{w}_{\mathrm{i}}\right)$ | $\mathrm{w}_{\mathrm{i}} \mathrm{y}_{\mathrm{i}}$ |
| :---: | :---: | :---: | :---: | :---: |
| 0 | 1.0 | 1.000000000 | 1 | 1.000000000 |
| 1 | 1.1 | 0.909090909 | 2 | 1.818181818 |
| 2 | 1.2 | 0.833333333 | 2 | 1.666666667 |
| 3 | 1.3 | 0.769230769 | 2 | 1.538461538 |
| 4 | 1.4 | 0.714285714 | 2 | 1.428571429 |
| 5 | 1.5 | 0.666666667 | 2 | 1.333333333 |
| 6 | 1.6 | 0.625000000 | 2 | 1.250000000 |
| 7 | 1.7 | 0.588235294 | 2 | 1.176470588 |
| 8 | 1.8 | 0.555555556 | 2 | 1.111111111 |
| 9 | 1.9 | 0.526315789 | 2 | 1.052631579 |
| 10 | 2.0 | 0.500000000 | 1 | 0.500000000 |

$1 ?^{2}(1 / \mathrm{x}) \mathrm{dx}=(0.05)(13.875428063)=0.693771403$

The value of $\ln 2$ is rounded to nine decimal places and we have seen that midpoint approximation produces a more accurate result than the trapezoidal approximation. Hence we can conclude that,
If $f$ be a continuous on $[a, b]$ and let $\left|\mathrm{E}_{\mathrm{M}}\right|$ and $\left|\mathrm{E}_{\mathrm{T}}\right|$ be the absolute errors that result from the midpoint and trapezoidal approximations of a ? ${ }^{\mathrm{b}} \mathrm{f}(\mathrm{x}) \mathrm{dx}$ using n subintervals.
a) If the graph of $f$ is either concave up or concave down on $(a, b)$, then $\left|E_{M}\right|<\left|E_{T}\right|$, which means that the error from the midpoint approximation is less than from the trapezoidal approximation.
b) If the graph of ' f ' is concave down on ( $\mathrm{a}, \mathrm{b}$ ) then $\mathrm{Tn}<\mathrm{a}$ ? $\mathrm{f}(\mathrm{x}) \mathrm{dx}<\mathrm{Mn}$
c) If the graph of ' f ' is concave up on ( $\mathrm{a}, \mathrm{b}$ ), then $\mathrm{Mn}_{\mathrm{n}}<\mathrm{a}$ ? ${ }^{\mathrm{b}} \mathrm{f}(\mathrm{x}) \mathrm{dx}<\mathrm{Tn}$

## Simpson's Rule

Simpson's Rule is given by
$S_{2 n}=\underline{1}\left(2 M_{n}+T_{n}\right)$
3

$$
=\frac{1}{3\left(\frac{b-a}{2 n}\right)\left[y_{0}+4 y_{1}+2 y_{2}+4 y_{3}+2 y_{4}+\ldots . .2 y_{n-2}+4 y_{2 n-1}+y_{2 n}\right]}
$$

Where $M_{n}=\left(\frac{b-a}{2 n}\right)\left[2 y_{1}+2 y_{3}+\ldots \ldots .+2 y_{2 n-1}\right]$

$$
\begin{aligned}
& \mathrm{T}_{\mathrm{n}}=\left(\frac{\mathrm{b}-\mathrm{a}}{2 \mathrm{n}}\right)^{\left[y_{0}+2 y_{2}+2 y_{4}+\ldots \ldots . .+2 y_{2 n-2}+y_{2 n}\right]} \\
& \text { Also a } \int^{\mathrm{b}} \mathrm{f}(\mathrm{x}) \mathrm{dx} \approx \mathrm{~S}_{2 \mathrm{n}} \\
& \text { We denote the absolute error in this approximation by }
\end{aligned}
$$

$$
\left|E_{S}\right|=\left|a \int^{b} f(x) d x-S_{2!1}\right|
$$

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## Reference links:

- http://en.wikipedia.org/wiki/Riemann_sum
- http://en.wikipedia.org/wiki/Trapezoidal_rule
- http:/ /en.wikipedia.org/wiki/Simpson's_rule
- http://www.purplemath.com/modules/numeric.htm


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