TAYLOR SERIES

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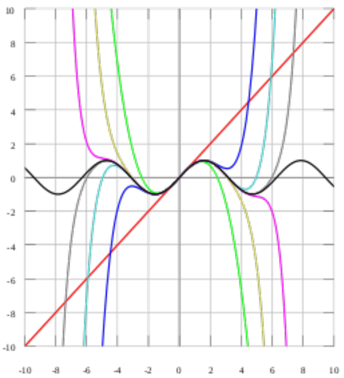
Introduction



If ‘f’ has derivatives of all orders at x_0, then we call the series

$$\Sigma \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\ldots\ldots\ldots+\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k+\ldots\ldots$$

The Taylor series for ‘f’ about $x=x_0$.
In the special case where $x_0=0$, this series becomes

$$\Sigma \frac{f^{(k)}(0)}{k!}x^k = f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\ldots\ldots\ldots+\frac{f^{(k)}(0)}{k!}x^k+\ldots\ldots\ldots\ldots\ldots$$

which is called as the Maclaurin series of ‘f’.
Note that the nth Maclaurin and Taylor polynomials are the nth partial sums for the corresponding Maclaurin and Taylor series.
Example: Find the Maclaurin series for 1/(1-x)
Solution: We know that the nth Maclaurin polynomial for 1/(1-x) is
$p_n(x)$ = ?x^k = 1+x+x^2+………+x^k (n=0, 1, 2………….)
Thus, the Maclaurin series for 1/(1-x) is
?x^k= 1+x+x^2+x^3+…….+x^k+…………..
Example: Find the Taylor series for 1/x about x=1
Solution: We know that the nth Taylor polynomial for 1/x about x=1 is
?$(-1)^k$(x-1)k=1-(x-1)+$(x-1)^2$-$(x-1)^3$+….+$(-1)^n(x-1)^n$
Thus the Taylor series for 1/x about x=1 is
?$(-1)^k$(x-1)k=1-(x-1)+$(x-1)^2$-$(x-1)^3$+……..+$(-1)^k(x-1)^k$+………

Power Series in x

Maclaurin and Taylor series differ from the series that we have discussed above that their terms are not merely constants, but instead involve a variable. These are examples of power series which is defined as
If c_0, c_1, c_2, ……. Are constants and x is a variable then a series of the form
?ckxk= $c_0+c_1x+c_2x^2$+………..+c_kx^k+……………….. is called a power series in x
Here are some examples

i) $\Sigma x^k = 1 + x + x^2 + x^3 + \ldots\ldots\ldots\ldots\ldots\ldots$

ii) $\Sigma \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots\ldots\ldots$

iii) $\Sigma \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots\ldots$

Indeed every Maclaurin series is a power series in x.

Radius and interval of convergence

If a numerical value is substituted for x in a power series $?c_k x^k$, then the resulting series of numbers may either converge or diverge. This leads to the problem of determining the set of x-values for which a given power series converges; this is called its convergence set.

For any a power series in x, exactly one of the following is true:

a) The series converges only for x=0

b) The series converges absolutely (and hence converges) for all real values of x

c) The series converges absolutely (and hence converges) for all x in some finite open interval (-R, R) and diverge if x<-R or x>R. At either of the values x=R or x=-R, the series may converge absolutely, converge conditionally or diverge, depending on the particular series.

This theorem states that the convergence set for a power series in x is always as interval centered at x=0 (possibly just the value x=0 itself or possibly infinite). For this reason, the convergence set of a power series in x is called the interval of convergence. In the case where the convergence set is the single value x=0 we say that the series has radius of convergence 0, in the case where the convergence set is (-?, +?) we say that the series has radius of convergence +? and in the case where the convergence set extends between –R and R we say that the series has radius of convergence R.

Diverges Diverges [Radius of convergence R=0]

0

Converges [Radius of convergence $R=+\infty$]

0

Diverges Converges Diverges [Radius of convergence R]

0

The usual procedure for finding the interval of convergence of a power series is to apply the ratio test for absolute convergence. The following example illustrates how this works.

Example: Find the interval of convergence and radius of convergence of the following power series $?x^k$ and $?x^k/k!$

Solution:

$?x^k$

We apply the ratio test for absolute convergence. We have

$$\rho = \lim_{k\to\infty} \left|\frac{u^{k+1}}{u^k}\right| = \lim_{k\to\infty} \left|\frac{x^{k+1}}{x^k}\right| = \lim_{k\to\infty} |x| = |x|$$

So the series converges absolutely if ? = |x| <1 and diverges if ?=|x|>1. The test is inconclusive if |x|=1 (x=1 or x=-1), which means that we will have to investigate convergence at these values separately. At these values the series becomes

$?1^k$ = 1 + 1+ 1+ 1+ 1+………… [x=1]

$?(-1)^k$ = 1-1+1-1+1-1+……..[x=-1]

Both of which diverge, thus, the interval of convergence for the given power series is (-1, 1) and the radius of convergence is R=1.

$?x^k/k!$

Applying the ratio test for absolute convergence, we obtain

$$\rho=\lim_{k\to\infty}\left|\frac{u^{k+1}}{u^k}\right|=\lim_{k\to\infty}\left|\frac{x^{k+1}}{(k+1)!}\,\frac{k!}{x^k}\right|=\lim_{k\to\infty}\left|\frac{x}{k+1}\right|=0$$

Power Series in x-x0

?$c_k(x-x_0)^k = c_0+c_1(x-x_0)+c_2(x-x_0)^2+\ldots..+c_k(x-x_0)^k+\ldots\ldots$ is called a power series in $x-x_0$.
For a power series ?$c_k(x-x_0)^k$, exactly one of the following statements is true:

a) The series converges only for $x=x_0$.

b) The series converges absolutely and hence converges for all real values of x.

c) The series converges and hence converges for all x in some finite open interval (x_0-R, x_0+R) and diverges is $x<x_0-R$ or $x>x_0+R$. At either of the values $x=x_0-R$ or $x=x_0+R$, the series may converge absolutely, converge conditionally or diverge depending on the particular series.

It follows from the above statements that the set of values for which a power series in $x-x_0$ converges is always an interval centered at $x=x_0$; we call this the interval of convergence. In part (a) the interval of convergence reduces to the single value $x=x_0$ in which case we say that the series has radius of convergence R=0; in part (b) the interval of convergence is infinite, in which case we say that the series has radius of convergence R=+?; and in part (c) the interval extends between x_0-R and x_0+R, in which case we say that the series has radius of convergence R.

Now try it yourself! Should you still need any help, click here to schedule live online session with e Tutor!

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Reference Links:

- http://en.wikipedia.org/wiki/Taylor_series
- http://www.intmath.com/series-expansion/2-maclaurin-series.php
- http://en.wikipedia.org/wiki/Power_series
- http://en.wikipedia.org/wiki/Radius_of_convergence

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