VOLUME OF A SOLID WITH KNOWN CROSS SECTION

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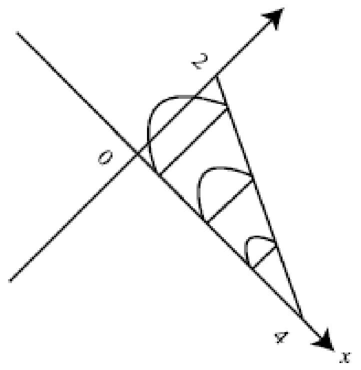
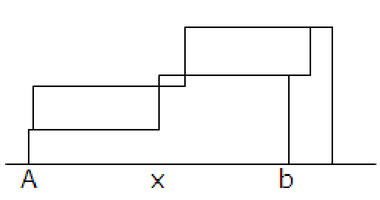
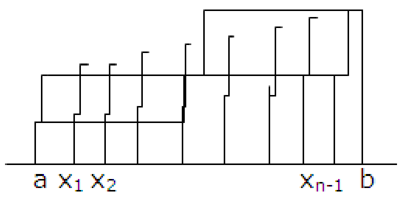
Introduction



We have already learned to find the area of a plane region bounded by two curves which is obtained by integrating the length of a general cross section over an appropriate interval. Here we will see that the same basic principle can be used to find volumes of certain three dimensional solids.

Let S be a solid that extends along the x-axis and is bounded on the left and right, respectively, by the planes that are perpendicular to the x-axis at x=a and x=b. We are finding the volume V of the solid, assuming that its cross-sectional area A(x) is known at each x in the interval [a, b].
To solve this problem we divide the interval [a, b] into n subintervals, which has the effect of dividing the solid into n slabs [fig (ii)]





If we assume that the width of the k^{th} slab is $?x_k$, then the volume of the slab can be approximated by the volume of a right cylinder of width (height) $?x_k$ and cross-sectional area $A(x_k^*)$, where x_k^* is a number in the k^{th} subinterval. Adding these approximations yields the following Riemann sum that approximates the volume V:

$$V ? ?A(x_k^*)?x_k$$

Taking the limit as n increases and the widths of the subintervals approach zero yields the definite integral

I. $$V = \lim_{max?x_k \quad 0} ?A(x_k^*)?x_k = {}_a?^b A(x)dx$$

We can conclude the result in the following way,

Volume formula

Let S be a solid bounded by two parallel planes perpendicular to the x-axis at x=a and x=b. If, for each x in [a, b] the cross-sectional area of S perpendicular to the x-axis is A(x), then the volume of the solid is,

$V= {}_a?^b A(x)\,dx$ provided A(x) is integrable.

Volume Formula

Let S be a solid bounded by two parallel planes perpendicular to the y-axis at y=c and y=d. If, for each y in [c, d], the cross-sectional area of S perpendicular to the y-axis is A(y), then the volume of the solid is,

$V = {}_c?^d A(y)\,dy$ provided A(y) is integrable.

In words, these formulas states that, “The volume of a solid can be obtained by integrating the cross-sectional area from one end of the solid to the other".

Example: Find the formula for the volume of a right pyramid whose altitude is h and whose base is a square with sides of length a.

Solution: We introduce a rectangular coordinate system in which the y-axis passes through apex and is perpendicular to the base, and the x-axis passes through the base and is parallel to a side of the base.

At any 'y' in the interval [0, h] on the y-axis, the cross section perpendicular to the y-axis is a square. If 's' denotes the length of a side of this square, then by similar triangles.

$$\frac{\frac{1}{2} s}{\frac{1}{2} a} = \frac{h-y}{h}$$

$s = (a/h)(h-y)$

Thus the area $A(y)$ of the cross section at y is

$A(y) = s^2 = (a^2/h^2)(h-y)^2$

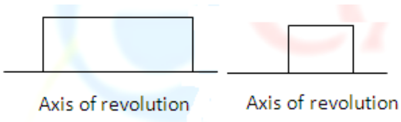
$V = {}_0\int^h A(y)dy = {}_0\int^h (a^2/h^2)(h-y)^2 = (a^2/h^2)\,{}_0\int^h (h-y)^2 dy$

$= (a^2/h^2)[-1/3(h-y)^3]_0^h = 1/3a^2h$

Thus the volume is 1/3 of the area of the base times the altitude.

Solids of revolution

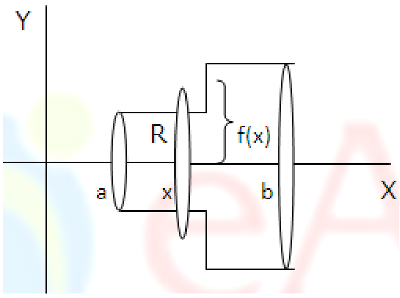
A solid of revolution is a solid that is generated by revolving a plane region about a line that lies in the same plane as the region; the line is called the axis of revolution.



Volume of a solid of revolution

Let f be continuous and non-negative on [a, b] and let R be the region that is bounded by y=f(x), below by the x-axis, and on the sides by the lines x=a and x=b, then the volume of the solid or revolution that is generated by revolving the region R about the x-axis is given by

$$V = {}_a?^b\ ?[f(x)]^2 dx$$
$$= {}_a?^b\ ?\ y^2\ dx$$
$$= ?\ {}_a?^b y^2\ dx$$



Example: Find the volume of a paraboloid of revolution formed by revolving the parabola $y^2=8x$ about the x-axis from x=0 to x=6
Solution: The equation of the parabola is $y^2=8x$,
Hence volume = $_0?^6\ ?y^2dx$
= $?_0?^6\ 8x\ dx$
= $8?[x^2/2]_0^6$
= $8?\ x\ ½\ [6^2-0^2]$
=4?x36
= 144?cubic units.

Volume by cylindrical shells

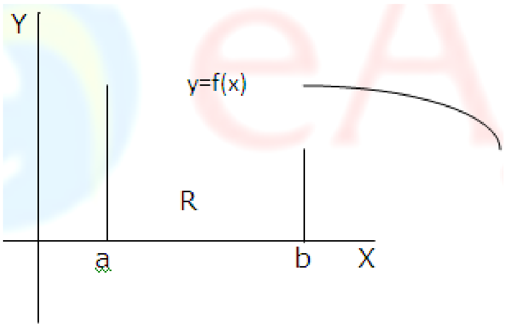
A cylindrical shell is a solid enclosed by two concentric right circular cylinders. The volume V of a cylindrical shell with inner radius r_1, outer radius r_2, and height h can be written as

V = (area of cross section).height

Let f be continuous and non-negative on [a, b] and let R be the region that is bounded above by y=f(x) below by the x-axis, and on the sides by the lines x=a and x=b. Then the volume V of the solid revolution that is generated by revolving the region R about the y-axis is given by

$V =_a?^b\ 2?xf(x)dx$

A.



Example: Use cylindrical shells to find the volume of the solid generated when the region R in the first quadrant enclosed between y=x and $y=x^2$ is revolved about the y-axis.
Solution: $V=\ _0?^1 2?x(x-x^2)dx = 2?\ _0?^1\ (x^2-x^3)dx$
= 2? [(1/3) – (1/4)]
= ?/6 cubic units.

Now try it yourself! Should you still need any help, click here to schedule live online session with e Tutor!

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Reference Links:

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