L'HOSPITAL'S RULE

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Introduction

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

The limit lim [f(x)/g(x)] is, in general, equal, to the limit of the numerator x a divided by the limit of the denominator. But when lim f(x) and lim g(x) are both zero, the quotient takes the form 0/0 which is meaningless. But lim [f(x)/g(x)] may not be meaningless, which mean that it may exist. In this article, we shall consider how to obtain the limiting values in such and similar other cases.

L'Hospital's Rule for the form 0/0: Suppose that f and g are differentiable functions on an open interval containing x=a, except possibly at x=a, and that

$$\lim_{x \to a} f(x) = 0 \text{ and } \lim_{x \to a} g(x) = 0$$

If lim[f'(x)/g'(x)] has a finite limit or if this limit is +? or -?, then x a

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Moreover, this statement is also true in the case of a limit as x -a-, x - a+, x - -? or as x- +?

Note that in L'Hospital's rule the numerator and denominator are differentiated separately, which is not the same as differentiating f(x)/g(x).

The following steps are used while solving problems:

Step I: Check that the limit of f(x)/g(x) is an indeterminate form. If it is not, then L'Hospital's rule cannot be used.

Step II: Differentiate f and g separately.

Step III: Find the limit of f'(x)/g'(x). If this limit is finite, +?, or -?, then it is equal to the limit of f(x)/g(x).

Example: Evaluate $\lim_{x \to 0} \frac{\sin x}{x}$ by L'Hospital's rule

Solution: $\lim_{x \to 0} \frac{\sin x}{x}$ [0/0 form]

$= \lim_{x \to 0} \frac{\cos x}{1}$ [Applying L'Hospital's rule]

$= 1/1$

$= 1$

L'Hospital's Rule for the form ?/?

Suppose that f and g are differentiable functions on an open interval containing x=a, except possibly at x=a, and that

$$\lim_{x \to a} f(x) = \infty \text{ and } \lim_{x \to a} g(x) = \infty$$

If lim [f'(x)/g'(x)] has a finite limit, or if this limit is +? or -? then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Moreover, this statement is also true in the case of a limit as x a-, x a+, x -? or as x +?

Example: Apply L’Hospital’s rule and evaluate the following

$$\lim_{x \to +\infty} \frac{x}{e^x}$$

Solution: The numerator and denominator both have a limt of +?, so we have an indeterminate form of type ?/?. Applying L’Hospital’s rule,

$$\lim_{x \to +\infty} \frac{x}{e^x} = \lim_{x \to +\infty} \frac{1}{e^x} = 0$$

Example: Evaluate $\lim_{x \to 0^+} \frac{\ln x}{\csc x}$

Example: Evaluate lim lnx
x 0+ cscx

Solution: The numerator has a limit of -? and the denominator has a limit of +?, so we have an indeterminate form of type ?/?.
Applying L’Hospital’s rule

$$\lim_{x \to 0^+} \frac{\ln x}{\csc x} = \lim_{x \to 0^+} \frac{1/x}{-\csc x \cot x} \quad \ldots\ldots\ldots\ldots(1)$$

This last limit is again an indeterminate form of type ?/?. Moreover, any additional applications of L’Hospital’s rule will yield powers of 1/x in the numerator and expressions involving cscx and cotx in the denominator; thus, repeated application of L’Hospital’s rule simply produces new indeterminate forms. We must try something else. The equation (1) can be written as,

$$\lim_{x \to 0^+} \left(-\frac{\sin x}{x} \tan x\right) = -\lim_{x \to 0^+} \frac{\sin x}{x} . \lim_{x \to 0^+} \tan x = -(1)(0) = 0$$

Thus $\lim_{x \to 0+} \frac{\ln x}{\csc x} = 0$

Indeterminate forms

There are some more other indeterminate forms other that 0/0 and ?/? forms. We can discuss about them now.
In general, the limit of an expression that has one of the forms f(x)/g(x), f(x).g(x), f(x)g(x), f(x)-g(x), f(x)+g(x) is called indeterminate form if the limits of f(x) and g(x) individually exert conflicting influences on the limit of the entire expression.

For example $\lim_{x \to 0+} x \ln x$

It is an indeterminate form of type 0.? because the limit of the first factor is 0 and the limit of the second factor is -? and these influences work together to produce a limit of -? for the product.
Indeterminate forms of 0.? can sometimes be evaluated by rewriting the product as a ratio and then applying L’Hospital’s rule for indeterminate forms of type 0/0 or ?/?.
Expressions of the form ?-?, 0^0, $?^0$ and $1^?$ are all indeterminate forms.

Cauchy Mean Value Theorem

Let f and g be continuous on [a, b] and differentiable on (a, b). Suppose that g’(x)?0 for x?(a, b). Then there exists c?(a, b) such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Examples:

$$\lim_{x \to 0^+} \frac{\log(\sin x)}{\log x} = \lim_{x \to 0^+} \frac{\cos x/\sin x}{1/x} = \lim_{x \to 0^+} \frac{\cos x}{1} \frac{\sin x}{x} = \cos 0 \times 1 = 1 \quad [\infty/\infty]$$

Example:

$$\text{Example: } \lim_{x \to 0^+} x\log x = \lim_{x \to 0^+} \frac{\log x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = 0 \qquad [0.\infty \text{ form}]$$

Example: Using Cauchy Mean Value Theorem, show that $1-(x^2/2!) < \cos x$ for x?0

Solution: Apply Cauchy Mean Value Theorem $f(x) = 1-\cos x$ and $g(x)=x^2/2$. We get $\frac{(1-\cos x)}{(x^2/2)} = \frac{\sin c}{c} < 1$ for some c between 0 and x

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Reference Links:

- http://en.wikipedia.org/wiki/L'H%C3%B4pital's_rule
- http://en.wikipedia.org/wiki/Differentiable_function
- http://en.wikipedia.org/wiki/Indeterminate_form
- http://en.wikibooks.org/wiki/Calculus/Mean_Value_Theorem

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