LOCAL LINEAR APPROXIMATION

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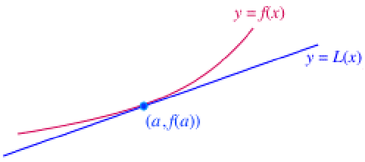
Introduction



In this section we will learn how derivatives can be used to approximate nonlinear functions by simpler linear functions. We will also define the differentials dy and dx and use them to interpret the derivative dy/dx as a ratio of differentials.

Let a function ‘f’ is differentiable at x0 and recall that the equation of the tangent line to the graph of the function ‘f’ through P(x_0, f(x_0)) is y=f(x_0)+f’(x_0)(x-x_0). Since this line closely approximates the graph of ‘f’ for values of x near xo, if follows that

f(x) ? f(x_0) + f’(x_0)(x-x_0) …………………………..(1)

Provided x is close to x_0. We call (1) the local linear approximation of ‘f’ at x_0. Furthermore, it can be shown that (1) is actually the best linear approximation of ‘f’ near x_0 in the sense that any other linear function will fail to give as good as approximation to f for values of x very close to x_0. An alternative version of this formula can be obtained by letting ?x= x-x_0 in which case (1) can be expressed as

f(x_0+?x) ? f(x_0) + f’(x_0)?x

In practice we use the following method for local linear approximation

1) We are given some value to calculate say f(x), and the calculation is difficult.
2) We see that there is a nearby point x_0, where the calculation of both f(x_0) and f(x_0)’ is relatively easy.
3) We approximate f(x) by f(x_0) + f(x_0)’(x-x_0)

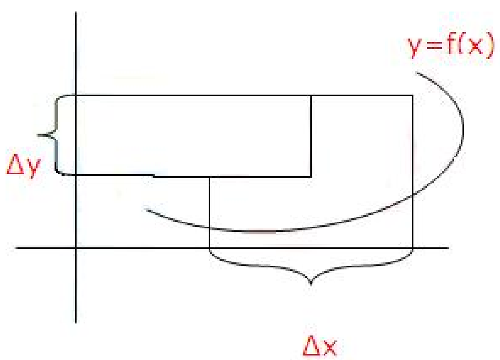
Example: Find the local linear approximation of f(x) = ?x at x_0=1

Solution: Given f(x) = ?x, f’(x) = 1/(2?x)

The local linear approximation of ?x at x_0=1 is

?x ? ?1 + ½ (x-1) = ½ (x+1)

Approximating Changes – Differentials



The second use of derivative is to approximate small changes in a function. Start at point x and move a small distance given by the independent variable ?x. Then define ?y to be the corresponding change in the value of y=f(x). We can see that ?y=f(x+?x)-f(x)

y+?y = f(x+?x)
?y = f(x+?x)-y
= f(x+?x)-f(x)

Let dx be an arbitrary variable and define dy to be f'(x)dx. Then the ratio of dy to dx is the derivative dy/dx. dx and dy are called the differentials.

Example: Find the differential dy when $y = x^{18}$ when x=1

Solution: $y=x^{18}$

$dy = 18x^{17}dx$

= 18dx since $1^{17} = 1$

Error Propagation in Applications

In applications, small errors invariably occur in measured quantities. When these quantities are used in computations, those errors are propagated in turn to the computed quantities. For example, suppose that in an application the variables x and y are related by a function y=f(x). If x_a is the actual value of x, and it is measured to be x_0, then we define the difference $dx=x_0-x_a$ to be the error in measurement of x. If the error is positive, the measured value is larger than the actual value and if the error is negative the measured value is smaller than the actual value. Since y is determined from x by the function y=f(x), the true value of y is $f(x_a)$ and the value of y computed from the measured value of x is $f(x_0)$. The propagated error in the computed value of y is defined to be $f(x_0) – f(x_a)$. If the propagated error is positive, the calculated value of y will be too large and if this error is negative, the calculated value of y will be too small.

The local linear approximation (1) becomes

$f(x_a)$? $f(x_0) +f'(x_0)(x_a-x_0)$

? $f(x_0) – f'(x_0) (x_0-x_a)$

? $f(x_0) – f'(x_0)$ dx

Example: Suppose that the side of a square is measured with a ruler to be 10 inches with a measurement error of at most ±1/32 of an inch. Use a differential to estimate the error in the computed area of the square.

Solution: The side of a square x and the area of the square y are related by the equation $y=x^2$

Since dy=2xdx and if x=10 then dy=20dx

To say that the measurement error is at most ±1/32 of an inch means that the measurement error $dx = x_0-x_a$ satisfies the inequalities -1/32?dx?1/32. Multiplying each term by 20 yields the equivalent inequalities

20(-1/32)? dy ?20(1/32)

-5/8 ?dy ?5/8

Since we are using the differential dy to approximate the propagated error, we estimate this propagated error to be between -5/8 and 5/8 of a square inch. Hence we estimate the propagated error to be at most ±5/8 of a square inch.

Relative error and Percentage error

The ratio of the error in some measured or calculated quantity to the true value of the quantity is called the relative error of the measurement or calculation. When expressed as a percentage, the relative error is called the percentage error.

For example, suppose that the side of a square is measured to be 10 inches, but the actual length of the side is 9.98 inches, the relative error in this measurement is 0.02/9.98?0.002004008.

Now try it yourself! Should you still need any help,click here to schedule live online session with e Tutor!

Reference Links:

- http://en.wikipedia.org/wiki/Linear_approximation
- http://www.buzzle.com/articles/relative-error.html
- http://en.wikipedia.org/wiki/Derivative

- http://en.wikipedia.org/wiki/Variable_(mathematics)

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