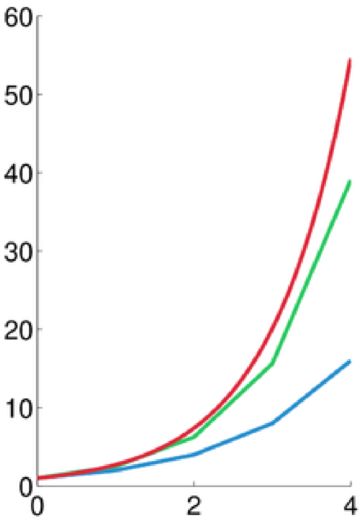
NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS USING EULER'S METHOD

Created: Thursday, 17 November 2011 10:27 | Published: Thursday, 17 November 2011 10:27 | Written by Super User | Print

Introduction



In this article our objective is to develop a method for approximating the solution of an initial-value problem of the form

$$y' = f(x, y), \quad y(x_0) = y_0$$

We will not attempt to approximate y(x) for all values of x; rather, we will choose some small increment ?x and focus on approximating the values of y(x) at a succession of x-values spaced ?x units apart, starting from x0. We will denote these x-values by $x_1=x_0+?x$, $x_2=x_1+?x$, $x_3=x_2+?x$, $x_4=x_3+?x$, ……… and we will denote the approximations of y(x) at these points by $y_1?y(x_1)$, $y_2?y(x_2)$, $y_3?y(x_3)$ …………
The technique that we will describe for obtaining these approximations is calledEuler's Method.

Euler's Method

To approximate the solution of the initial-value problem

$$y'=f(x, y), \quad y(x_0) = y_0$$

Proceed as follows

Step 1: Choose a nonzero number ?x to serve as an increment or step size along the x-axis, and let

$x_1=x_0+?x$, $x_2= x_1+?x$, $x_3 = x_2+?x$, ………………………

Step 2: Compute successively

$$y_1 = y_0 + f(x_0, y_0)?x$$

$$y_2 = y_1 + f(x_1, y_1)?x$$

$$y_3 = y_2 + f(x_2, y_2)?x$$

…………………………………..

$$y_{n+1} = y_n + f(x_n, y_n)?x$$

The numbers y_1, y_2, y_3 ……… in these equations are the approximations of $y(x_1)$, $y(x_2)$, $y(x_3)$, …………………………..
Example: Use Euler's Method with a step size of 0.1 to make a table of approximate values of the solution of the initial-value problem

$$y'=y-x, \; y(0)=2 \text{ over the interval } 0?x?1.$$

Solution: In this problem we have f(x, y)=y-x, $x0=0$ and $y0=2$. Moreover, since the step size is 0.1, the x-values at which the approximate values will be obtained are

$x_1=0.1$, $x_2=0.2$, $x_3=0.3$, ……………………….., $x_9=0.9$, $x_{10}=1$

The first three approximations are

$y_1=y_0+f(x_0, y_0)$?x = 2+(2-0)(0.1)=2.2

$y_2=y_1+f(x_1, y_1)$?x = 2.2+(2.2-0.1)(0.1) = 2.41

$y_3=y_2+f(x_2, y_2)$?x=2.41+(2.41-0.2)(0.1) = 2.631

Here is a way of organizing all 10 approximations rounded to five decimal places.

EULER's METHOD FOR $y'=y-x$, $y(0)=2$ with $\Delta x=0.1$

n	x_n	y_n	$f(x_n, y_n)\Delta x$	$y_{n+1}=y_n + f(x_n, y_n)\Delta x$
0	0	2.00000	0.20000	2.20000
1	0.1	2.20000	0.21000	2.41000
2	0.2	2.41000	0.22100	2.63100
3	0.3	2.63100	0.23310	2.86410
4	0.4	2.86410	0.24641	3.11051
5	0.5	3.11051	0.26105	3.37156
6	0.6	3.37156	0.27716	3.64872
7	0.7	3.64872	0.29487	3.94359
8	0.8	3.94359	0.31436	4.25795
9	0.9	4.25795	0.33579	4.59374
10	1.0	4.59374	–	–

Observe that each entry in the last column becomes the next entry in the third column.

Accuracy of Euler's Method

We can compare the approximate values of y(x) produced by Euler's Method with decimal approximation of the exact values.

The absolute error is calculated as

Absolute error = |exact value – approximation| and the percentage error as

$$\text{Percentage error} = \frac{|\text{exact value} - \text{approximation}|}{|\text{exact value}|} \times 100\%$$

The absolute error and percentage error of the above problem is shown below.

x	EXACT SOLUTION	EULER APPROXIMATION	ABSOLUTE ERROR	PERCENTAGE ERROR
0.0	2.00000	2.00000	0.00000	0.00
0.1	2.20517	2.20000	0.00517	0.23
0.2	2.42140	2.41000	0.01140	0.47
0.3	2.64986	2.63100	0.01886	0.71
0.4	2.89182	2.86410	0.02772	0.96
0.5	3.14872	3.11051	0.03821	1.21
0.6	3.42212	3.37156	0.05056	1.48
0.7	3.71375	3.64872	0.06503	1.75
0.8	4.02554	3.94359	0.08195	2.04
0.9	4.35960	4.25795	0.10165	2.33
1.0	4.71828	4.59374	0.12454	2.64

Exponential Growth and Decay Models

A quantity y=y(t) is said to be an exponential growth model it is increases at a rate that is proportional to the amount of the quantity present, and it is said to have an exponential decay model if ir decreases at a rate that is proportional to the amount of the quantity present. Thus, for an exponential growth model, the quantity y(t) satisfies an equation of the form

$$dy/dt = ky \ (k>0)$$

and for an exponential decay model, the quantity y(t) satisfies an equation of the form

$$dy/dt = -ky \ (k>0)$$

The constant k is called the growth constant or the decay constant, as appropriate.

The above written two equations are first order linear equations, since they can be rewritten as

$$(dy/dt)-ky = 0 \text{ and } (dy/dt) + ky=0$$

To illustrate how these equations can be solved, suppose that a quantity y=y(t) has an exponential growth model and we know the amount of the quantity at some point in time, $y=y_0$ when t=0. Thus a general formula for y(t) can be obtained by solving the initial value problem

$$(dy/dt)-ky=0, \ y(0)=y0$$

Multiplying by integrating factor, $\mu=e^{-kt}$ yields $d/dt(e^{-kt} y)=0$ and then integrating with respect to 't' yields

$$e^{-kt} y =C \text{ or } y=Ce^{kt}$$

the initial condition implies that $y=y_0$ when t=0, from which it follows that $C=y_0$. Thus, the solution of the initial-value problem is

$$y = y_0e^{kt}$$

We leave it for you to show that if y=y(t) has an exponential decay model, and if $y(0)=y_0$, then

$$y = y_0e^{-kt}$$

Doubling time and half-life

If a quantity y has an exponential growth model, then the time required for the original size to double is called the doubling time, and if y has an exponential decay model, then the time required for the original size to reduce by half is called the half-life. As it turns out, doubling time and half-life depend only on the growth or decay rate and not on the amount present initially. To see why this is so, suppose that y=y(t) has an exponential growth model $y=y_0e^{kt}$ and let T denote the amount of time required for y to double in size. Thus, at time t=T the value of y will be $2y_0$ and hence the above equation becomes $2y_0 = y_0e^{kT}$ or $e^{kT}=2$

Hence T = (1/k)ln2

Now try it yourself! Should you still need any help, click here to schedule live online session with e Tutor!

About eAge Tutoring:

eAgeTutor.com is the premium online tutoring provider. Using materials developed by highly qualified educators and leading content developers, a team of top-notch software experts, and a group of passionate educators, eAgeTutor works to ensure the success and satisfaction of all of its students.

Contact us today to learn more about our tutoring programs and discuss how we can help make the dreams of the student in your life come true!

Reference Links:

- http://en.wikipedia.org/wiki/Initial_value_problem
- http://en.wikipedia.org/wiki/Euler_method
- http://en.wikipedia.org/wiki/Exponential_growth
- http://en.wikipedia.org/wiki/Half-life

Category:ROOT