RIEMANN SUM AND TRAPEZOIDAL RULE

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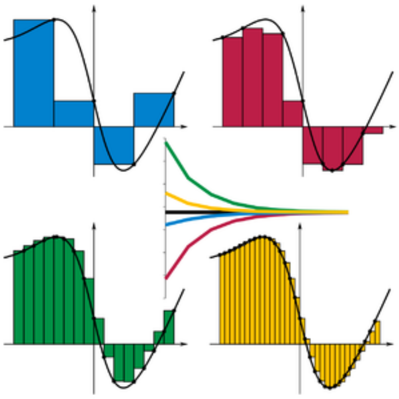
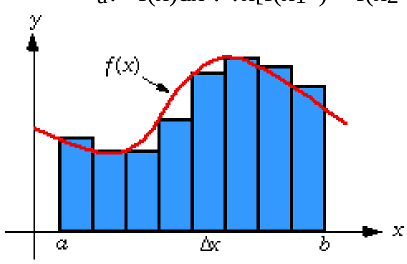
Riemann Sum Approximation



The definite integral of a continuous function ‘f’ over an interval [a, b] is computed as $_a?^b$ f(x)dx = lim ?f(x_k*)?x, where the sum that appears on the right side is called Riemann sum. In this formula, the interval [a, b] is divided into n subintervals of width ?x = (b-a)/n, and x_k* denotes an arbitrary point in the k^{th} sub-interval It follows that as n increases the Riemann sum will eventually be a good approximation to the integral, which we denote by writing

$a?^b$ f(x)dx ? ?f(x_k*)?x

$a?^b$ f(x)dx ? ?x[f(x_1^*) + f(x_2^*) + …….+f(x_n^*)]



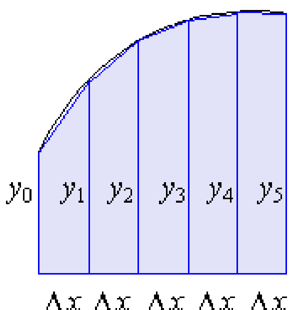
Here we denote the values of ‘f’ at the endpoints of the subintervals by

y_0=f(a), y_1=f(x_1), y_2=f(x_2), ………, y_{n-1}=f(x_{n-1}), y_n=f(b) and we will denote the values of f at the midpoints of the subintervals by y_{m1}, y_{m2}, ……y_{mn}

Trapezoidal Approximation

The left-hand and right hand endpoint approximations are rarely used in applications; however, if we take the average of the left-hand and right hand endpoint approximations, we obtain a result, called the trapezoidal approximation, which is commonly used as,

$a?^b$ f(x)?x ? (b-a)/2n [y_0 + 2y_1 + …..+ 2y_{n-1} + y_n]



represents the area under f(x) over [a, b]. Geometrically, the trapezoidal approximation formula results if we approximate this area by the sum of the trapezoidal areas as shown in the figure

Left end point Approximation: The formula for evaluating left end point approximation is given by

$_a?^b$ f(x)dx = (b-a)/n [$y_0 + y_1 + \ldots\ldots\ldots. + y_{n-1}$]

Right Endpoint Approximation: The formula for evaluating right end point approximation is given by

$_a?^b$ f(x)dx = (b-a)/n [$y_1 + y_2 + \ldots\ldots\ldots\ldots + y_n$]

Mid-Point Approximation: The formula for evaluating midpoint approximation is given by

$_a?^b$ f(x)dx = (b-a)/n [$ym_1 + ym_2 + \ldots\ldots\ldots. + ymn$] where $m_1, m_2 \ldots\ldots m_n$ represents the mid values.

Example: Use Trapezoidal rule to approximate $_0?^?$sinx dx using n=10 sub intervals

Solution: a=0, b=? n=10 and f(x)=sinx, (b-a)/n = ?/10

i	0	1	2	3	....	10
x_i	0	п/10	2п/10	3п/10	....	10п/10
y_i	sin(0)	sin(п/10)	sin(2п/10)	sin(3п/10)	....	sin(10п/10)

$_0?^?$ sinx dx = (?/20)[$y_0 + 2y_1 + 2y_2 + \ldots\ldots\ldots\ldots + 2y_{n-1} + y_n$]

= (?/20) [sin(0) + 2sin(?/10) + 2sin(2?/10) + ……..sin(10?/10)

= 1.983523538

Comparison of the Midpoint and Trapezoidal Approximations

The table below shows the comparison between midpoint and trapezoidal approximations for the function ln 2 =$_1?^2$(1/x)dx with n=10 subdivisions

Midpoint Approximation

i	Midpoint (m_i)	y_{mi} = f(m_i)=1/m_i
1	1.05	0.952380952
2	1.15	0.869565217
3	1.25	0.800000000
4	1.35	0.740740741
5	1.45	0.689655172
6	1.55	0.645161290
7	1.65	0.606060606
8	1.75	0.571428571
9	1.85	0.540540541
10	1.95	0.512820513
		6.928353603

$_1?^2$ (1/x)dx = (0.1)(6.928353603) = 0.692835360

Trapezoidal Approximation

i	Endpoint(x_i)	$y_i=f(x_i)=1/x_i$	Multiplier (w_i)	w_iy_i
0	1.0	1.000000000	1	1.000000000
1	1.1	0.909090909	2	1.818181818
2	1.2	0.833333333	2	1.666666667
3	1.3	0.769230769	2	1.538461538
4	1.4	0.714285714	2	1.428571429
5	1.5	0.666666667	2	1.333333333
6	1.6	0.625000000	2	1.250000000
7	1.7	0.588235294	2	1.176470588
8	1.8	0.555555556	2	1.111111111
9	1.9	0.526315789	2	1.052631579
10	2.0	0.500000000	1	0.500000000
				13.875428063

$\int_1^2 (1/x)dx = (0.05)(13.875428063) = 0.693771403$

The value of ln 2 is rounded to nine decimal places and we have seen that midpoint approximation produces a more accurate result than the trapezoidal approximation. Hence we can conclude that,

If f be a continuous on [a, b] and let $|E_M|$ and $|E_T|$ be the absolute errors that result from the midpoint and trapezoidal approximations of $\int_a^b f(x)dx$ using n subintervals.

a) If the graph of f is either concave up or concave down on (a, b), then $|E_M|<|E_T|$, which means that the error from the midpoint approximation is less than from the trapezoidal approximation.

b) If the graph of ‘f’ is concave down on (a, b) then $T_n < \int_a^b f(x)dx < M_n$

c) If the graph of ‘f’ is concave up on (a,b), then $M_n < \int_a^b f(x)dx < T_n$

Simpson’s Rule

Simpson’s Rule is given by

$$S_{2n} = \frac{1}{3}(2M_n + T_n)$$

$$= \frac{1}{3}\left(\frac{b-a}{2n}\right)[y_0+4y_1+2y_2+4y_3+2y_4+ \ldots\ldots 2y_{n-2}+4y_{2n-1}+y_{2n}]$$

Where $M_n = \left(\frac{b-a}{2n}\right)[2y_1 + 2y_3 + \ldots\ldots + 2y_{2n-1}]$

$T_n = \left(\frac{b-a}{2n}\right)[y_0 + 2y_2 + 2y_4 + \ldots\ldots + 2y_{2n-2} + y_{2n}]$

Also $\int_a^b f(x)dx \approx S_{2n}$

We denote the absolute error in this approximation by

$$|E_S| = \left|\int_a^b f(x)dx - S_{2n}\right|$$

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Reference links:

- http://en.wikipedia.org/wiki/Riemann_sum
- http://en.wikipedia.org/wiki/Trapezoidal_rule
- http:/ /en.wikipedia.org/wiki/Simpson's_rule
- http://www.purplemath.com/modules/numeric.htm

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