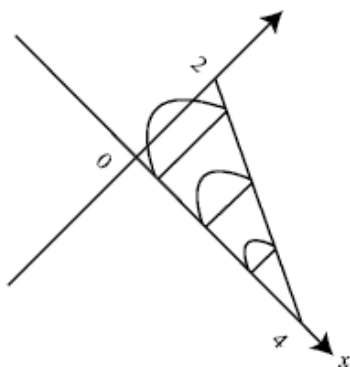


# VOLUME OF A SOLID WITH KNOWN CROSS SECTION

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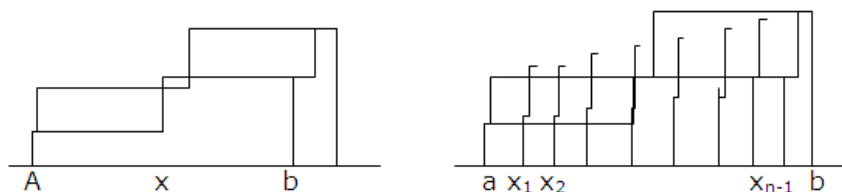
## Introduction



We have already learned to find the [area](#) of a plane region bounded by two curves which is obtained by integrating the length of a general cross section over an appropriate interval. Here we will see that the same basic principle can be used to find volumes of certain three dimensional solids.

Let  $S$  be a solid that extends along the  $x$ -axis and is bounded on the left and right, respectively, by the planes that are perpendicular to the  $x$ -axis at  $x=a$  and  $x=b$ . We are finding the [volume](#)  $V$  of the solid, assuming that its cross-sectional area  $A(x)$  is known at each  $x$  in the interval  $[a, b]$ .

To solve this problem we divide the interval  $[a, b]$  into  $n$  subintervals, which has the effect of dividing the solid into  $n$  slabs [fig (ii)]



If we assume that the width of the  $k^{\text{th}}$  slab is  $\Delta x_k$ , then the volume of the slab can be approximated by the volume of a right cylinder of width (height)  $\Delta x_k$  and cross-sectional area  $A(x_k^*)$ , where  $x_k^*$  is a number in the  $k^{\text{th}}$  subinterval. Adding these approximations yields the following [Riemann sum](#) that approximates the volume  $V$ :

$$V \approx \sum_{k=1}^n A(x_k^*) \Delta x_k$$

Taking the limit as  $n$  increases and the widths of the subintervals approach zero yields the definite integral

$$I. \quad V = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n A(x_k^*) \Delta x_k = \int_a^b A(x) dx$$

We can conclude the result in the following way,

### Volume formula

Let  $S$  be a solid bounded by two parallel planes perpendicular to the  $x$ -axis at  $x=a$  and  $x=b$ . If, for each  $x$  in  $[a, b]$  the cross-sectional area of  $S$  perpendicular to the  $x$ -axis is  $A(x)$ , then the volume of the solid is,

$$V = \int_a^b A(x) dx \text{ provided } A(x) \text{ is integrable.}$$

### Volume Formula

Let  $S$  be a solid bounded by two parallel planes perpendicular to the  $y$ -axis at  $y=c$  and  $y=d$ . If, for each  $y$  in  $[c, d]$ , the cross-sectional area of  $S$  perpendicular to the  $y$ -axis is  $A(y)$ , then the volume of the solid is,

$$V = \int_c^d A(y) dy \text{ provided } A(y) \text{ is integrable.}$$

In words, these formulas states that, "The volume of a solid can be obtained by integrating the cross-sectional area from

one end of the solid to the other”.

Example: Find the formula for the volume of a right pyramid whose altitude is  $h$  and whose base is a square with sides of length  $a$ .

Solution: We introduce a rectangular coordinate system in which the  $y$ -axis passes through apex and is perpendicular to the base, and the  $x$ -axis passes through the base and is parallel to a side of the base.

At any ' $y$ ' in the interval  $[0, h]$  on the  $y$ -axis, the cross section perpendicular to the  $y$ -axis is a square. If ' $s$ ' denotes the length of a side of this square, then by similar triangles.

$$\frac{1}{2} s = \frac{h-y}{h} a$$

$$s = \frac{a}{h}(h-y)$$

Thus the area  $A(y)$  of the cross section at  $y$  is

$$A(y) = s^2 = \left(\frac{a}{h}\right)^2 (h-y)^2$$

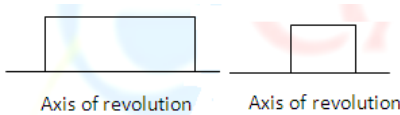
$$V = \int_0^h A(y) dy = \int_0^h \left(\frac{a^2}{h^2}\right) (h-y)^2 dy = \left(\frac{a^2}{h^2}\right) \int_0^h (h-y)^2 dy$$

$$= \left(\frac{a^2}{h^2}\right) \left[-\frac{1}{3}(h-y)^3\right]_0^h = \frac{1}{3} a^2 h$$

Thus the volume is  $1/3$  of the area of the base times the altitude.

## Solids of revolution

A **solid of revolution** is a solid that is generated by revolving a plane region about a line that lies in the same plane as the region; the line is called the axis of revolution.



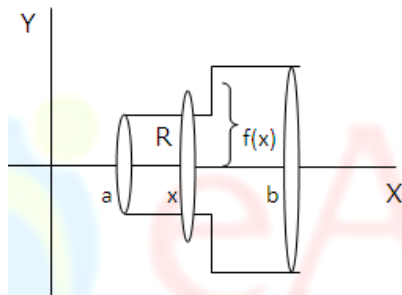
## Volume of a solid of revolution

Let  $f$  be continuous and non-negative on  $[a, b]$  and let  $R$  be the region that is bounded by  $y=f(x)$ , below by the  $x$ -axis, and on the sides by the lines  $x=a$  and  $x=b$ , then the volume of the solid or revolution that is generated by revolving the region  $R$  about the  $x$ -axis is given by

$$V = \int_a^b \pi [f(x)]^2 dx$$

$$= \pi \int_a^b y^2 dx$$

$$= \pi \int_a^b y^2 dx$$



Example: Find the volume of a paraboloid of revolution formed by revolving the parabola  $y^2 = 8x$  about the x-axis from  $x=0$  to  $x=6$

Solution: The equation of the parabola is  $y^2 = 8x$ ,

$$\begin{aligned} \text{Hence volume} &= \int_0^6 \pi y^2 dx \\ &= \int_0^6 \pi 8x dx \\ &= 8\pi \left[ \frac{x^2}{2} \right]_0^6 \\ &= 8\pi \times \frac{1}{2} [6^2 - 0^2] \\ &= 4\pi \times 36 \\ &= 144\pi \text{ cubic units.} \end{aligned}$$

## Volume by cylindrical shells

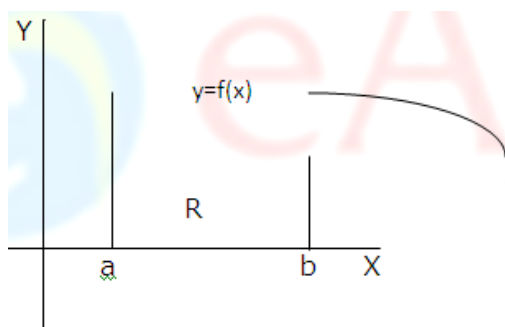
A [cylindrical shell](#) is a solid enclosed by two concentric right circular cylinders. The volume  $V$  of a cylindrical shell with inner radius  $r_1$ , outer radius  $r_2$ , and height  $h$  can be written as

$$V = (\text{area of cross section}) \cdot \text{height}$$

Let  $f$  be continuous and non-negative on  $[a, b]$  and let  $R$  be the region that is bounded above by  $y=f(x)$  below by the x-axis, and on the sides by the lines  $x=a$  and  $x=b$ . Then the volume  $V$  of the solid revolution that is generated by revolving the region  $R$  about the y-axis is given by

$$V = \pi \int_a^b 2xf(x)dx$$

A.



Example: Use cylindrical shells to find the volume of the solid generated when the region  $R$  in the first quadrant enclosed between  $y=x$  and  $y=x^2$  is revolved about the y-axis.

$$\begin{aligned} \text{Solution: } V &= \pi \int_0^1 2x(x-x^2)dx = 2\pi \int_0^1 (x^2 - x^3)dx \\ &= 2\pi \left[ \frac{1}{3} - \frac{1}{4} \right] \\ &= \frac{2\pi}{6} \text{ cubic units.} \end{aligned}$$

Now try it yourself! Should you still need any help, [click here](#) to schedule live online session with e Tutor!

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### Reference Links:

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[http://www.cliffsnotes.com/study\\_guide/Volumes-of-Solids-with-Known-Cross-Sections.topicArticleId-39909,articleId-39906.html](http://www.cliffsnotes.com/study_guide/Volumes-of-Solids-with-Known-Cross-Sections.topicArticleId-39909,articleId-39906.html)

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