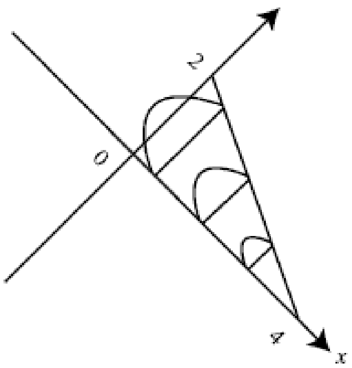
VOLUME OF A SOLID WITH KNOWN CROSS SECTION

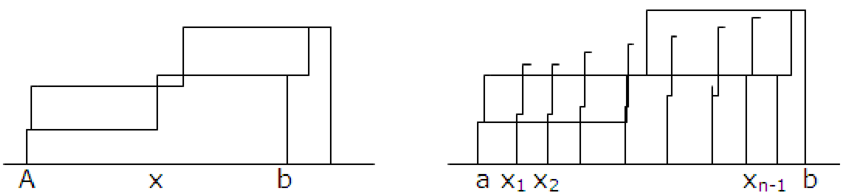
Created: Saturday, 19 November 2011 10:21 | Published: Saturday, 19 November 2011 10:21 | Written by Super User | Print

Introduction



We have already learned to find the area of a plane region bounded by two curves which is obtained by integrating the length of a general cross section over an appropriate interval. Here we will see that the same basic principle can be used to find volumes of certain three dimensional solids.

Let S be a solid that extends along the x-axis and is bounded on the left and right, respectively, by the planes that are perpendicular to the x-axis at x=a and x=b. We are finding the volume V of the solid, assuming that its cross-sectional area A(x) is known at each x in the interval [a, b].
To solve this problem we divide the interval [a, b] into n subintervals, which has the effect of dividing the solid into n slabs [fig (ii)]



If we assume that the width of the k^{th} slab is $?x_k$, then the volume of the slab can be approximated by the volume of a right cylinder of width (height) $?x_k$ and cross-sectional area $A(x_k*)$, where x_k* is a number in the k^{th} subinterval. Adding these approximations yields the following Riemann sum that approximates the volume V:

$$V ? ?A(x_k*)?x_k$$

Taking the limit as n increases and the widths of the subintervals approach zero yields the definite integral

I. $$V = \lim_{max?x_k \quad 0} ?A(x_k*)?x_k = {}_a?^b A(x)dx$$

We can conclude the result in the following way,

Volume formula

Let S be a solid bounded by two parallel planes perpendicular to the x-axis at x=a and x=b. If, for each x in [a, b] the cross-sectional area of S perpendicular to the x-axis is A(x), then the volume of the solid is,

$$V= {}_a?^b A(x)\,dx$$ provided A(x) is integrable.

Volume Formula

Let S be a solid bounded by two parallel planes perpendicular to the y-axis at y=c and y=d. If, for each y in [c, d], the cross-sectional area of S perpendicular to the y-axis is A(y), then the volume of the solid is,

$$V = {}_c?^d A(y)\,dy$$ provided A(y) is integrable.

In words, these formulas states that, “The volume of a solid can be obtained by integrating the cross-sectional area from one end of the solid to the other".

Example: Find the formula for the volume of a right pyramid whose altitude is h and whose base is a square with sides of length a.

Solution: We introduce a rectangular coordinate system in which the y-axis passes through apex and is perpendicular to the base, and the x-axis passes through the base and is parallel to a side of the base.

At any 'y' in the interval [0, h] on the y-axis, the cross section perpendicular to the y-axis is a square. If 's' denotes the length of a side of this square, then by similar triangles.

$$\frac{\tfrac{1}{2} s}{\tfrac{1}{2} a} = \frac{h-y}{h}$$

$s = (a/h)(h-y)$

Thus the area A(y) of the cross section at y is

$A(y) = s^2 = (a^2/h^2)(h-y)^2$

$V = {}_0\int^h A(y)dy = {}_0\int^h (a^2/h^2)(h-y)^2 = (a^2/h^2)\,{}_0\int^h (h-y)^2 dy$

$= (a^2/h^2)[-1/3(h-y)^3]_0^h = 1/3a^2h$

Thus the volume is 1/3 of the area of the base times the altitude.

Solids of revolution

Asolid of revolution is a solid that is generated by revolving a plane region about a line that lies in the same plane as the region; the line is called the axis of revolution.

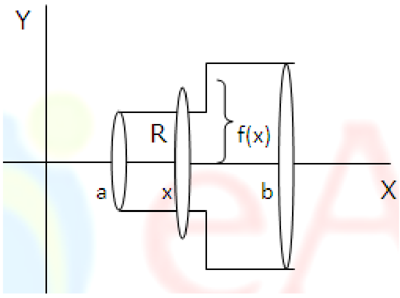

Axis of revolution


Axis of revolution

Volume of a solid of revolution

Let f be continuous and non-negative on [a, b] and let R be the region that is bounded by y=f(x), below by the x-axis, and on the sides by the lines x=a and x=b, then the volume of the solid or revolution that is generated by revolving the region R about the x-axis is given by

V= $_a?^b$?$[f(x)]^2$dx
= $a?^b$? y^2 dx
= ? $_a?^b y^2$ dx

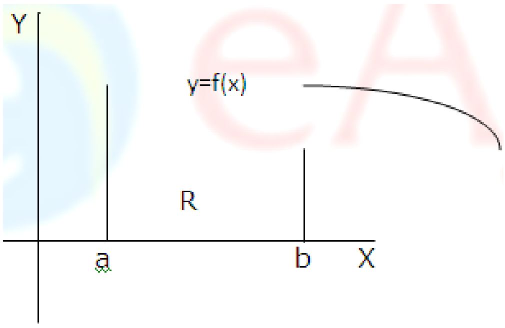


Example: Find the volume of a paraboloid of revolution formed by revolving the parabola y^2=8x about the x-axis from x=0 to x=6
Solution: The equation of the parabola is y^2=8x,
Hence volume = $_0?^6$ $?y^2$dx
= $?_0?^6$ 8x dx
= $8?[x^2/2]_0^6$
= 8? x ½ $[6^2-0^2]$
=4?x36
= 144?cubic units.

Volume by cylindrical shells

A cylindrical shell is a solid enclosed by two concentric right circular cylinders. The volume V of a cylindrical shell with inner radius r_1, outer radius r_2, and height h can be written as
V = (area of cross section).height
Let f be continuous and non-negative on [a, b] and let R be the region that is bounded above by y=f(x) below by the x-axis, and on the sides by the lines x=a and x=b. Then the volume V of the solid revolution that is generated by revolving the region R about the y-axis is given by
V =$_a?^b$ 2?xf(x)dx

A.



Example: Use cylindrical shells to find the volume of the solid generated when the region R in the first quadrant enclosed between y=x and y=x^2 is revolved about the y-axis.
Solution: V= $_0?^1$ $2?x(x-x^2)$dx = 2? $_0?^1$ (x^2-x^3)dx
= 2? [(1/3) – (1/4)]
= ?/6 cubic units.

Now try it yourself! Should you still need any help, click here to schedule live online session with e Tutor!

About eAge Tutoring:

eAgeTutor.com is the premium online tutoring provider. Using materials developed by highly qualified educators and leading content developers, a team of top-notch software experts, and a group of passionate educators, eAgeTutor works to ensure the success and satisfaction of all of its students.

Contact us today to learn more about our tutoring programs and discuss how we can help make the dreams of the student in your life come true!

Reference Links:

http://www.intmath.com/applications-integration/2-area-under-curve.php

http://www.cliffsnotes.com/study_guide/Volumes-of-Solids-with-Known-Cross-Sections.topicArticleId-39909,articleId-39906.html http://en.wikipedia.org/wiki/Solid_of_revolution

http://en.wikipedia.org/wiki/Shell_integration

http://en.wikipedia.org/wiki/Riemann_sum

Category:ROOT